Journal of Computational and Applied Mathematics 32 (1990) 159–167
North-Holland

On the relationship between generalised continued fractions and G-continued fractions

Paul LEVRIE
Katholieke Industriële Hogeschool Antwerpen, Salesianenlaan 1A, B-2710 Hoboken, Belgium, and Departement Computerwetenschappen, Katholieke Universiteit Leuven, Celestijnenlaan 200A, B-3030 Heverlee, Belgium

Received 24 September 1989

Abstract: In this paper the connection between generalised continued fractions (de Bruin (1974)) and G-continued fractions (Levrie (1988)) is studied. This connection is used to prove a convergence theorem for generalised continued fractions and to accelerate the convergence of generalised continued fractions associated with a class of linear recurrence relations of Poincaré-type.

Keywords: Recurrence relation, generalised continued fraction, convergence acceleration.

1. Introduction

In this paper we discuss the relationship between two different generalisations of continued fractions, both associated with a linear recurrence relation of arbitrary order.

The first type of generalisation is the so-called n-fraction or generalised continued fraction (GCF). GCFs are used in the study of simultaneous rational approximation of functions using rational functions with a common denominator [1]. They can also be used for the computation of certain nondominant solutions of linear recurrence relations (see [3,7,8]).

The second type of generalisation of a continued fraction is the G-continued fraction introduced in [6,10]. G-continued fractions are used for the calculation of minimal solutions of linear recurrence relations (i.e., solutions for which the Miller algorithm converges [6]). The method is related to Gautschi's continued fraction algorithm for second-order recurrence relations [2].

In 1976 it was shown by Zahar [11] that there exists a close connection between the convergence of the G-continued fraction associated with a linear recurrence relation and the convergence of the GCF associated with its adjoint equation. In this paper we take a closer look at this result and we will show that the value of a GCF may be calculated from the value of the G-continued fraction associated with the adjoint recurrence relation. Furthermore, it is shown that under some mild conditions the convergence of the G-continued fraction implies the convergence of the GCF. These results can, for instance, be used to derive new convergence theorems for GCFs from known convergence theorems for G-continued fractions.

In the rest of this section we will give the necessary definitions and notations concerning GCFs and G-continued fractions. In Section 2 we prove that the approximants of a GCF associated with a linear recurrence relation may be calculated from the approximants of the G-continued fraction associated with the adjoint equation. We also prove a convergence result for GCFs. In Section 3 we look at some consequences of this relationship for GCFs. First of all we prove a convergence theorem for GCFs related to Pringsheim's theorem for ordinary continued fractions. Then we look at how convergence acceleration for G-continued fractions can be used to accelerate the convergence of n-fractions.

We consider a pth order recurrence relation of the form

$$c_0(n)y_{n+p} + c_1(n)y_{n+p-1} + \cdots + c_{p-1}(n)y_{n+1} + c_p(n)y_n = 0, \quad n = 0, 1, \ldots, \tag{1}$$

with $c_i(n) \in \mathbb{C}$, $c_0(n)c_p(n) \neq 0$ for all i and n. If $f_n^{(1)}, f_n^{(2)}, \ldots, f_n^{(p-1)}$ are solutions of this recurrence relation, we shall use the notation $E_n(f^{(1)}, \ldots, f^{(p-1)})$ for the determinant of the matrix

$$\begin{pmatrix} f_n^{(1)} & f_n^{(2)} & \cdots & f_n^{(p-1)} \\ \vdots & \vdots & & \vdots \\ f_{n+p-2}^{(1)} & f_{n+p-2}^{(2)} & \cdots & f_{n+p-2}^{(p-1)} \end{pmatrix}.$$

The $(p-1)$-fraction or generalised continued fraction (see [1]) associated with (1), denoted by

$$\underset{n=0}{\overset{\infty}{\mathrm{K}}} \begin{pmatrix} -c_p(n)/c_0(n) \\ \vdots \\ -c_1(n)/c_0(n) \end{pmatrix}, \tag{2}$$

is given by the sequence of approximants $\{C_k^{(1)}/C_k^{(p)}, \ldots, C_k^{(p-1)}/C_k^{(p)}\}_{k=p}^{\infty}$ where the numerators and denominators satisfy the recurrence relation (1) with initial values

$$C_n^{(i)} = \delta_{n,i-1}, \quad i = 1, \ldots, p, \quad n = 0, 1, \ldots, p-1.$$

The GCF is said to converge in $\mathbb{C}^{p-1}$ if the following limit exists in $\mathbb{C}^{p-1}$:

$$\{b_{p-1}(0), \ldots, b_1(0)\} = \lim_{k \to \infty} \{C_k^{(1)}/C_k^{(p)}, \ldots, C_k^{(p-1)}/C_k^{(p)}\}. \tag{3}$$

The Nth approximant of the GCF, $\{C_{N+p}^{(1)}/C_{N+p}^{(p)}, \ldots, C_{N+p}^{(p-1)}/C_{N+p}^{(p)}\}$, may be calculated using the following algorithm: let

$$\begin{aligned} &b_1^N(N+1) = \cdots = b_{p-1}^N(N+1) = 0, \\ &b_h^N(k) = \frac{c_{h+1}(k) - c_0(k)b_{h+1}^N(k+1)}{c_1(k) - c_0(k)b_1^N(k+1)}, \quad h = 1, \ldots, p-2, \\ &b_{p-1}^N(k) = \frac{c_p(k)}{c_1(k) - c_0(k)b_1^N(k+1)}, \end{aligned} \tag{4}$$

for $k = N, N-1, \ldots, 0$. Then we have

$$b_h^N(0) = \frac{C_{N+p}^{(p-h)}}{C_{N+p}^{(p)}}, \quad h = 1, \ldots, p-1. \tag{5}$$

The G-continued fraction associated with the recurrence relation (1) is given by the sequence of approximants $\{-T_k^{(2)}/T_k^{(1)}\}_{k=2}^{\infty}$ where $T_n^{(h)}$, $h=1,\ldots,p$, is defined by

$$T_n^{(h)} = E_n(C^{(1)},\ldots,C^{(h-1)},\ C^{(h+1)},\ldots,C^{(p)})$$

(see [6]). The G-continued fraction converges if the following limit exists:

$$B(0) = -\lim_{k\to\infty}\frac{T_k^{(2)}}{T_k^{(1)}}. \tag{6}$$

The Nth approximant $B^N(0) = -T_{N+2}^{(2)}/T_{N+2}^{(1)}$ may be calculated using the algorithm

$$B^N(N+i) = 0, \quad i=1,\ldots,p-1,$$

$$B^N(k) = -\frac{c_p(k)}{\displaystyle\sum_{i=1}^{p} c_{p-i}(k)\left(\prod_{h=1}^{i-1} B^N(k+h)\right)}, \quad k=N,\ldots,0. \tag{7}$$

The G-continued fraction above is denoted by

$$\mathop{\mathrm{K}}_{n=0}^{\infty}\left(\frac{-c_p(n)/c_0(n)}{c_{p-1}(n)/c_0(n);\ldots;c_1(n)/c_0(n)}\right). \tag{8}$$

The jth tail of the G-continued fraction is then defined to be

$$t^{(j)} = \mathop{\mathrm{K}}_{n=j}^{\infty}\left(\frac{-c_p(n)/c_0(n)}{c_{p-1}(n)/c_0(n);\ldots;c_1(n)/c_0(n)}\right).$$

If the jth tail converges, then its value is denoted by $B(j)$.

2. Connection between the two types of generalised continued fractions

Let us consider the linear recurrence relation

$$a_p(n+1)y_{n+p} + a_{p-1}(n)y_{n+p-1} + \cdots + a_1(n-p+2)y_{n+1} + y_n = 0, \quad n=0,1,\ldots, \tag{9}$$

with $a_i(n)\in\mathbb{C}$ for all i and n, and $a_p(n)\neq 0$ for all n, and let $A_n'^{(i)}$, $i=1,\ldots,p$, be the solution of (9) with initial values given by

$$A_n'^{(i)} = \delta_{n,i-1}, \quad n=0,1,\ldots,p-1. \tag{10}$$

The adjoint equation of (9) is given by

$$y_{n+p} + a_1(n)y_{n+p-1} + \cdots + a_{p-1}(n)y_{n+1} + a_p(n)y_n = 0, \quad n=0,1,\ldots . \tag{11}$$

Let $A_n^{(i)}$ be the corresponding solutions of (11). It is possible to construct a fundamental system of solutions for (11) from the $A_n'^{(i)}$ [9]: let $U_n^{(h)}$ $(h=1,\ldots,p)$ be defined by

$$U_n^{(h)} = E_n(A'^{(1)},\ldots,A'^{(h-1)},\ A'^{(h+1)},\ldots,A'^{(p)});$$

then $y_n^{(h)}$, $h=1,\ldots,p$, defined by

$$y_n^{(h)} = (-1)^{np}U_n^{(h)}\prod_{j=0}^{n-1}a_p(j), \tag{12}$$

form a fundamental system of solutions for the recurrence relation (11). From the initial values (10) we deduce that

$$\begin{aligned} &U_n^{(h)}=0, \quad n=0,\dots,h-1,\\ &U_h^{(h)}=(-1)^{(p-1)(h-1)}\prod_{j=1}^{h-1}\frac{1}{a_p(j)}, \quad h=1,\dots,p-1,\\ &U_0^{(p)}=1, \qquad U_n^{(p)}=0, \quad n=1,\dots,p-1. \end{aligned} \tag{13}$$

Furthermore, if for $h=1,\dots,p-1$ we define $y_n^{(h)}=0$, $n=h-p+1,\dots,-1$, then, using (12), (13), we obtain p initial values for the $y_n^{(h)}$:

$$\begin{aligned} &y_n^{(h)}=0, \quad n=0,\dots,h-1,\\ &y_h^{(h)}=(-1)^{p+h-1}a_p(0), \quad h=1,\dots,p-1,\\ &y_0^{(p)}=1, \qquad y_n^{(p)}=0, \quad n=1,\dots,p-1, \end{aligned}$$

and we can calculate them for $n=h+1,\dots$ from

$$y_{n+p}+a_1(n)y_{n+p-1}+\cdots+a_{p-1}(n)y_{n+1}+a_p(n)y_n=0, \quad n=h-p+1,\dots .$$

Using this it is not difficult to prove that we have

$$\begin{aligned} &A_n^{(1)}=y_n^{(p)},\\ &A_n^{(i)}=\frac{(-1)^{p+i}}{a_p(0)}\left(y_n^{(i-1)}+\sum_{j=i}^{p-1}(-1)^{j-i+1}a_{j-i+1}(j-p)y_n^{(j)}\right), \quad i=2,\dots,p, \end{aligned} \tag{14}$$

for all $n\geqslant 0$.

The Nth approximant of the GCF associated with (11) is given by

$$\left\{b_{p-1}^N(0),\dots,b_1^N(0)\right\}=\left\{\frac{A_{N+p}^{(1)}}{A_{N+p}^{(p)}},\dots,\frac{A_{N+p}^{(p-1)}}{A_{N+p}^{(p)}}\right\}.$$

For the approximants of the G-continued fraction associated with (9) we have from [3]:

$$\prod_{h=0}^{j}B^{N+p-2}(h)=(-1)^{j+1}\frac{y_{N+p}^{(j+2)}}{y_{N+p}^{(1)}}, \quad j=0,\dots,p-2.$$

Hence, using (14) we find for $h=2,\dots,p-1$:

$$\begin{aligned} b_{p-h}^N(0)=\frac{A_{N+p}^{(h)}}{A_{N+p}^{(p)}}&=\frac{(-1)^{p+h}\left(y_{N+p}^{(h-1)}+\sum_{j=h}^{p-1}(-1)^{j-h+1}a_{j-h+1}(j-p)y_{N+p}^{(j)}\right)}{(-1)^{p+p}y_{N+p}^{(p-1)}}\\ &=\frac{(-1)^h y_{N+p}^{(h-1)}/y_{N+p}^{(1)}+(-1)^h\sum_{j=h}^{p-1}(-1)^{j-h+1}a_{j-h+1}(j-p)y_{N+p}^{(j)}/y_{N+p}^{(1)}}{(-1)^p y_{N+p}^{(p-1)}/y_{N+p}^{(1)}} \end{aligned}$$

$$= \frac{(-1)^{h-2} y_{N+p}^{(h-1)}/y_{N+p}^{(1)} + \sum_{j=h}^{p-1} (-1)^{j-1} a_{j-h+1}(j-p) y_{N+p}^{(j)}/y_{N+p}^{(1)}}{(-1)^{p-2} y_{N+p}^{(p-1)}/y_{N+p}^{(1)}}$$

$$= \frac{\prod_{j=0}^{h-3} B^{N+p-2}(j) + \sum_{j=h}^{p-1} a_{j-h+1}(j-p) \prod_{k=0}^{j-2} B^{N+p-2}(k)}{\prod_{j=0}^{p-3} B^{N+p-2}(j)}$$

and

$$b_{p-1}^{N}(0) = \frac{A_{N+p}^{(1)}}{A_{N+p}^{(p)}} = (-1)^{2p} a_p(0) \frac{y_{N+p}^{(p)}}{y_{N+p}^{(p-1)}} = -a_p(0) B^{N+p-2}(p-2).$$

Hence we have proved the following theorem.

Theorem 1. *The approximants of the GCF associated with* (11) *and the G-continued fraction associated with* (9) *are related by*

$$b_{p-h}^{N}(0) = \frac{\prod_{j=0}^{h-3} B^{N+p-2}(j) + \sum_{j=h}^{p-1} a_{j-h+1}(j-p) \prod_{k=0}^{j-2} B^{N+p-2}(k)}{\prod_{j=0}^{p-3} B^{N+p-2}(j)}, \quad h = 2, \ldots, p-1,$$

$$b_{p-1}^{N}(0) = -a_p(0) B^{N+p-2}(p-2).$$

We have the following convergence theorem.

Theorem 2. *If the G-continued fraction associated with* (9) *and its tails* $t^{(1)}, \ldots, t^{(p-2)}$ *converge, with* $B(0)B(1) \cdot \ldots \cdot B(p-3) \neq 0$, *then the* $(p-1)$*-fraction associated with* (11) *converges.*

Proof. This theorem is a consequence of two other theorems, one of them due to Van der Cruyssen [8], the other one given in [6]. In [8] Van der Cruyssen proves the following result. The GCF associated with (11) converges if the equation

$$a_p(n+p-1) y_{n+p} + a_{p-1}(n+p-2) y_{n+p-1} + \cdots + a_1(n) y_{n+1} + y_n = 0,$$
$$n = 0, 1, \ldots, \tag{15}$$

has a fundamental system of solutions $f_n^{(1)}, \ldots, f_n^{(p-1)}$, g_n with $g_0 \neq 0$ and for which

$$\lim_{n \to \infty} \frac{E_n(f^{(1)}, \ldots, f^{(h-1)}, g, f^{(h+1)}, \ldots, f^{(p-1)})}{E_n(f^{(1)}, \ldots, f^{(p-1)})} = 0 \tag{16}$$

for $h = 1, \ldots, p-1$.

In [6] we stated the following result. If the G-continued fraction associated with (9) and its tails $t^{(1)}, \ldots, t^{(p-2)}$ converge, then the recurrence relation (9) has a fundamental system of solutions $f_n'^{(1)},\dots,f_n'^{(p-1)}$, g_n' for which

$$\lim_{n\to\infty}\frac{E_n\left(f'^{(1)},\dots,f'^{(h-1)},g',f'^{(h+1)},\dots,f'^{(p-1)}\right)}{E_n\left(f'^{(1)},\dots,f'^{(p-1)}\right)}=0$$

and with $g_0'\neq 0$. Hence if we choose

$$f_n^{(1)}=f_{n+p-2}'^{(1)},\dots,f_n^{(p-1)}=f_{n+p-2}'^{(p-1)},\quad g_n=g_{n+p-2}'$$

for all $n\geqslant 0$, then (16) is satisfied. Furthermore, from [6] we have that $g_0=g_{p-2}'=B(p-3)\cdot B(p-2)\cdot\ldots\cdot B(0)g_0'\neq 0$.

This proves the theorem. □

Remark. We note that the condition $B(0)B(1)\cdot\ldots\cdot B(p-3)\neq 0$ is not a very strong one since there is a certain freedom of choice for the $a_i(k)$ with $k<0$.

3. Applications

3.1. A Pringsheim-like convergence theorem

From Theorem 2 and [4] we obtain the following variant of Pringsheim's theorem.

Theorem 3. *If the coefficients of the recurrence relation* (11) *satisfy the inequalities*

$$|a_1(n)|\geqslant 1+|a_2(n+1)|+|a_3(n+2)|+\cdots+|a_p(n+p-1)| \tag{17}$$

for all $n\geqslant -p+2$, where the $a_i(k)$ with $k<0$ may be chosen arbitrarily (*but satisfying* (17)), *then the GCF associated with* (11) *converges.*

Proof. In [4] we have proved that if the coefficients of the recurrence relation (9) satisfy the inequalities (17), then the G-continued fraction associated with (9) and all its tails converge. Furthermore, we have that $|B^N(k)|<1$ for all n and for all $k\leqslant N+p-1$. Since the $B^N(k)$ satisfy the nonlinear recurrence relation

$$B^N(k)=-\frac{1}{\sum_{i=1}^{p}a_i(k-p+i+1)\left(\prod_{h=1}^{i-1}B^N(k+h)\right)},\quad k=N,\dots,0, \tag{18}$$

it is easy to see that

$$|B^N(k)|\geqslant\frac{1}{\sum_{i=1}^{p}|a_i(k-p+i+1)|}$$

for all N and hence

$$|B(k)|\geqslant\frac{1}{\sum_{i=1}^{p}|a_i(k-p+i+1)|}>0.$$

The result then follows from Theorem 2. □

3.2. Convergence acceleration for n-fractions

It is immediately clear that we can use convergence acceleration methods for G-continued fractions to calculate the value of GCFs. In [6] we have discussed a convergence acceleration method for G-continued fractions for which the associated recurrence relation is of Poincaré-type. Let us assume that the coefficients of (11) satisfy $\lim_{k\to\infty} a_i(k) = a_i$ for all i with $a_p \neq 0$. Furthermore, let us assume that the roots $w_1, w_2, \ldots, w_p$ of the characteristic polynomial associated with (11)

$$x^p + a_1 x^{p-1} + a_2 x^{p-2} + \cdots + a_{p-1} x + a_p$$

are all different in modulus:

$$|w_1| > |w_2| > \cdots > |w_p|.$$

In this case the recurrence relation (9) is also of Poincaré-type and its characteristic polynomial has the roots $1/w_1, 1/w_2, \ldots, 1/w_p$. It was shown in [6] that the following algorithm

$$\begin{aligned} &\tilde{B}^N(N+i) = 1/w_1, \quad i = 1, \ldots, p-1, \\ &\tilde{B}^N(k) = -\frac{1}{\sum_{i=1}^{p} a_i(k-p+i+1)\left(\prod_{h=1}^{i-1} \tilde{B}^N(k+h)\right)}, \quad k = N, \ldots, 0, \end{aligned} \tag{19}$$

calculates approximations to $B(0)$: $\lim_{N\to\infty} \tilde{B}^N(0) = B(0)$. Furthermore, it converges faster than the algorithm (7) in the sense that

$$\lim_{N\to\infty} \frac{B(0) - \tilde{B}^N(0)}{B(0) - B^N(0)} = 0.$$

Let us consider an example: let the coefficients of the third-order recurrence relation (11) be given by:

$$\begin{cases} a_1(k) = -6.0 + 0.8^{k+1}, \\ a_2(k) = 11.75 + 0.8^{k+1}, \\ a_3(k) = -7.5. \end{cases} \tag{20}$$

Its characteristic polynomial is given by

$$x^3 - 6x^2 + 11.75\, x - 7.5,$$

and it has the zeros $w_1 = 2.5$, $w_2 = 2.0$, $w_3 = 1.5$. The two-fraction associated with this recurrence relation converges to the value $\{b_1(0) = -4.1856365261145,\ b_2(0) = 2.8278080774441\}$. In this case we get for the expressions in Theorem 1:

$$b_1^N(0) = \frac{1 + a_1(-1)B^{N+1}(0)}{B^{N+1}(0)} = \frac{1 - 5B^{N+1}(0)}{B^{N+1}(0)},$$

$$b_2^N(0) = -a_3(0)B^{N+1}(1) = 7.5\, B^{N+1}(1).$$

If instead of the B's we use the $\tilde{B}$'s, we obtain approximations $\tilde{b}_i^N(0)$ to $b_i(0)$, $i = 1, 2$,

$$\tilde{b}_1^N(0) = \frac{1 - 5\tilde{B}^{N+1}(0)}{\tilde{B}^{N+1}(0)}, \qquad \tilde{b}_2^N(0) = 7.5\, \tilde{B}^{N+1}(1),$$

for which

$$\lim_{N\to\infty} \frac{b_1(0)-\tilde{b}_1^N(0)}{b_1(0)-b_1^N(0)}=0, \qquad \lim_{N\to\infty} \frac{b_2(0)-\tilde{b}_2^N(0)}{b_2(0)-b_2^N(0)}=0.$$

We shall briefly look at yet another convergence acceleration method for G-continued fractions. Let us assume that the third-order recurrence relation

$$y_{n+3}+c_1(n)y_{n+2}+c_2(n)y_{n+1}+c_3(n)y_n=0, \quad n=0, 1,\dots, \tag{21}$$

is of Poincaré-type and that the roots v_1, v_2, v_3 of its characteristic equation satisfy $|v_1|>|v_2|>|v_3|$. Furthermore, let us assume that for the coefficients of the recurrence relation we have:

$$\lim_{k\to\infty} \frac{v_3^3+c_1(k)v_3^2+c_2(k)v_3+c_3(k)}{v_3^3+c_1(k-1)v_3^2+c_2(k-1)v_3+c_3(k-1)}=t.$$

Then the tails of the G-continued fraction associated with (21) satisfy:

$$B(k)-v_3\sim\epsilon_k=-\frac{v_3^3+c_1(k)v_3^2+c_2(k)v_3+c_3(k)}{c_2(k)+v_3(c_1(k)+v_3)+v_3t(c_1(k)+v_3+v_3t)}, \quad k\to\infty,$$

(see [5]). This means that $v_3+\epsilon_k$ is a better approximation for the tail $B(k)$ than v_3. (Note that since (21) corresponds to (9) we have for the coefficients

$$c_1(n)=\frac{a_2(n)}{a_3(n+1)}, \qquad c_2(n)=\frac{a_1(n-1)}{a_3(n+1)}, \qquad c_3(n)=\frac{1}{a_3(n+1)},$$

and also $v_3=1/w_1$.) So if instead of (19) we calculate

$$\hat{B}^N(N+i)=v_3+\epsilon_{N+i}$$
$$=-\frac{c_3(N+i)-v_3^2t(c_1(N+i)+v_3+v_3t)}{c_2(N+i)+v_3(c_1(N+i)+v_3)+v_3t(c_1(N+i)+v_3+v_3t)},$$
$$i=1, 2,$$
$$\hat{B}^N(k)=-\frac{c_3(k)}{c_2(k)+\hat{B}^N(k+1)(c_1(k)+\hat{B}^N(k+2))}, \quad k=N,\dots,0,$$

Table 1a

N	$b_1^N(0)$	$\tilde{b}_1^N(0)$	$\hat{b}_1^N(0)$
10	−5.5507756452	1.0787329721	−2.7900507390
20	−2.4689419056	−3.8533310213	−4.1670793098
30	−3.8508073502	−4.1785600543	−4.1856021305
40	−4.1417619483	−4.1855482791	−4.1856364810
50	−4.1807541049	−4.1856354992	−4.1856365261
60	−4.1851095743	−4.1856365143	−4.1856365261
70	−4.1855799065	−4.1856365260	−4.1856365261
80	−4.1856304461	−4.1856365261	−4.1856365261
90	−4.1856358733	−4.1856365261	−4.1856365261
100	−4.1856364560	−4.1856365261	−4.1856365261

Table 1b

N	$b_2^N(0)$	$\tilde{b}_2^N(0)$	$\hat{b}_2^N(0)$
10	4.0079837740	−1.7235539762	1.6212463200
20	1.3436052593	2.5405066638	2.8117640627
30	2.5383247517	2.8216899709	2.8277783400
40	2.7898754250	2.8277317817	2.8278080384
50	2.8235868816	2.8278071896	2.8278080774
60	2.8273524906	2.8278080672	2.8278080774
70	2.8277591258	2.8278080773	2.8278080774
80	2.8278028208	2.8278080774	2.8278080774
90	2.8278075130	2.8278080774	2.8278080774
100	2.8278080168	2.8278080774	2.8278080774

(with $v_3 = 0.4$ and $t = 0.8$) and

$$\hat{b}_1^N(0) = \frac{1 - 5\hat{B}^{N+1}(0)}{\hat{B}^{N+1}(0)}, \qquad \hat{b}_2^N(0) = 7.5\ \hat{B}^{N+1}(1),$$

we obtain approximations to the value of the GCF associated with (20). In Table 1 we have used these three methods to calculate the value of the GCF.

References

[1] M.G. de Bruin, Generalized continued fractions and a multidimensional Padé table, Doctoral Thesis, Amsterdam, 1974.

[2] W. Gautschi, Computational aspects of three-term recurrence relations, *SIAM Rev.* **9** (1967) 24–82.

[3] P. Levrie, Het numeriek oplossen van lineaire recursiebetrekkingen: een veralgemening van de kettingbreukmethode van Gautschi, Doctoral Thesis, Kath. Univ. Leuven, 1987.

[4] P. Levrie, Pringsheim's theorem revisited, *J. Comput. Appl. Math.* **25** (1) (1989) 93–104.

[5] P. Levrie, *G*-continued fractions and convergence acceleration in the solution of third-order linear recurrence relations of Poincaré-type, *Appl. Number. Math.*, submitted.

[6] P. Levrie and R. Piessens, Convergence acceleration for Miller's algorithm, in: A. Cuyt, Ed., *Nonlinear Numerical Methods and Rational Approximation* (Reidel, Dordrecht, 1988) 349–370.

[7] P. Van der Cruyssen, Linear difference equations and generalized continued fractions, *Computing* **22** (1979) 269–278.

[8] P. Van der Cruyssen, Properties and applications of generalized continued fractions, Doctoral Thesis, Univ. Antwerpen, 1982.

[9] J. Wimp, *Computation with Recurrence Relations* (Pitman, Boston, 1984).

[10] R.V.M. Zahar, Computational algorithms for linear difference equations, Doctoral Thesis, Purdue Univ., 1968.

[11] R.V.M. Zahar, A mathematical analysis of Miller's algorithm, *Numer. Math.* **27** (1977) 427–447.